at x_0. Its derivative of order $(n+2)$, the difference quotient, has a limiting value also, and is denoted thus: $Q^{n+2}_{0,0,n+2}$. Then we have

$$\text{(V)} \qquad \left.\frac{dQ^{n+1}_{1,n+1}(x)}{dx}\right|_{x=x_0} = \frac{Q^{n+2}_{0,0,n+2}}{n+2}.$$

Proceeding in this way one can determine the properties of $Q^{n+1}_{1,n+1}(x)$ from those of $f(x)$, and in that way can obtain the oscillation of the remainder for any interval. If the problem is to determine the derivative of $f(x)$, one differentiates in formula (III), and obtains the expression for the remainder in that case in terms of $Q^{n+1}_{1,n+1}(x)$ and its derivative, which are already known.

In this way, the difference quotient enables us to expand an arbitrary function, analytic or merely continuous, real or complex, at points to suit the peculiar needs of the problem. Moreover, in any special case, in which a simpler and less elastic method of expansion can be used, the expansion in terms of difference quotients reduces automatically to that method. The remainder can be expressed explicitly in terms of the difference quotient, not as in the case of a remainder in terms of a derivative at some unknown point, but taken at the points about which we are expanding and the one additional point x, the point at which we want the remainder.

A THEOREM IN THE GEOMETRY OF THE TRIANGLE.

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It is the purpose of this paper to state a general theorem of which the well-known Wallace (or Simson[1]) line theorem is a very special case; to point out the result of a polar reciprocation; to mention some special cases of the theorem; and to give an application to the geometry of the quadrilateral.

1. *Theorem.*[2] $A_1A_2A_3$ is a triangle, a_i the side opposite A_i, $(i = 1, 2, 3)$, P a point on the circumference of the circumscribed circle[3] C, Q any point in the plane. Let A_iQ cut the circle[3] at U_i; PU_i intersects a_i at L_i. Then L_1, L_2, L_3, Q are collinear on l. Let us call this line the "P-transversal of Q" with respect to the triangle $A_1A_2A_3$.

The theorem follows at once from Pascal's Theorem for, since $A_2A_3A_1U_1PU_2$, $A_1A_2A_3U_3PU_1$ are inscribed hexagons, L_1, L_2, Q and L_3, L_1, Q are collinear.

Cor. 1.—If a transversal cut the side a_i of a triangle at L_i, and P be any point on the circumcircle, and if PL_i cut the circle at U_i, then A_1U_1, A_2U_2, A_3U_3 are concurrent at Q, a point on the transversal.

Cor. 2.—If a transversal cut the side a_i of a triangle at L_i, if Q is any point

[1] For references, see this MONTHLY, 1916, p. 61.

[2] It would be strange if this simple theorem were new; yet the writer has not found it recorded. A special case is given in Hatton's *Projective Geometry*, 1913, Ex. p. 164.

[3] Or conic.

on the transversal, and if A_iQ cut the circumcircle at U_i, then U_1L_1, U_2L_2, U_3L_3 are concurrent at P, a point on the circumcircle.

2. Now let the polar reciprocal of the figure be found with respect to any circle having P for center (Fig. 1). The sides of the triangle $A_1A_2A_3$ or T reciprocate into the vertices of a triangle $A_1'A_2'A_3'$ or T', which is inversely similar to T; and A_1', A_2', A_3', P lie on a circle C'. For[1]

$$\measuredangle A_1'PA_3' = \measuredangle A_3A_2A_1 = \measuredangle A_3PA_1 = \measuredangle A_1'A_2'A_3'.$$

If $A_1'U_1'$ is the reciprocal of L_1, $\measuredangle A_3'A_1'U_1' = \measuredangle A_2PL_1 = \measuredangle A_2A_1U_1$. Similarly, if $A_2'U_2'$ is the reciprocal of L_2, $\measuredangle A_1'A_2'U_2' = \measuredangle A_3PL_2 = \measuredangle A_3A_2U_2$. Hence Q', the reciprocal of l, is the point in T' isogonally conjugate to the point which corresponds to Q in the inverse similarity of two triangles.

It is clear that A_iU_i reciprocates into L_i', the point where PU_i' intersects a_i'; and that Q reciprocates into l', the P-transversal of Q' with respect to T'.

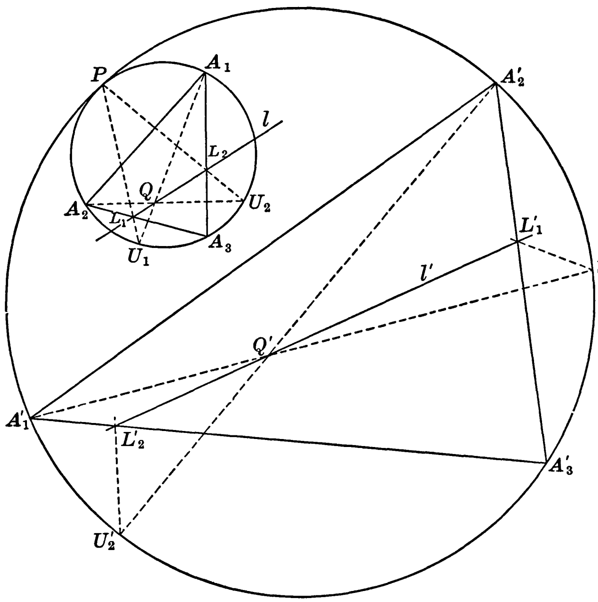

FIG. 1.

In particular, if Q is at the in-center or at one of the ex-centers of T, its reciprocal is the P-transversal of the in-center or the corresponding ex-center of T'.

3. If the point Q is at infinity,[2] it is easily shown by elementary geometry, remembering that A_1U_1, A_2U_2, A_3U_3 are parallel, that $\measuredangle PL_1A_2 = \measuredangle PL_2A_3 = \measuredangle PL_3A_1$. The line is in this case one of the family of lines described by

[1] The notation is that of R. A. Johnson for "directed angles" (this MONTHLY, 1917, p. 101), $\measuredangle BAC$ meaning the positive angle through which the line AB, taken as a whole, must be rotated, to coincide with the line AC, taken as a whole.

[2] This is the case given in Hatton, *loc. cit.*

Poncelet in his *Propriétés Projectives*, 1822, § 468, as generalizations of the Wallace line.[1]

Cor. If l is a transversal cutting the side a_i of a triangle in L_i and if A_iU_i is drawn parallel to it cutting the circumcircle at U_i, then L_1U_1, L_2U_2, L_3U_3 concur at a point P on the circumcircle. Moreover $\measuredangle\, PL_1A_2 = \measuredangle\, PL_2A_3 = \measuredangle\, PL_3A_1$.[2]

Now in the general case, $\measuredangle\, PL_1A_2 = \measuredangle\, Q'A_1'P$ and $\measuredangle\, PL_2A_3 = \measuredangle\, Q'A_2'P$. Hence, in the special case under consideration, $\measuredangle\, Q'A_1'P = \measuredangle\, Q'A_2'P$. Hence Q' is on the circle C', l' being in fact the chord PQ'.

Since the reciprocals of L_1, L_2, L_3 concur at a point on the circle C', the line l touches a parabola, focus P, which touches the sides of the triangle T. Hence Poncelet's lines envelope this parabola.[3]

In particular, if Q' is the point diametrically opposite to P in C', l will evidently be the pedal (or Wallace) line of the point P with respect to the triangle T. That is, if Q is at infinity in the direction of the directrix of the parabola of Steiner, l is the pedal line.

Since, in this last case, Q' and the circumcenter of T' are collinear with P, their reciprocals, the pedal line of P and the P-transversal of H, where H is the orthocenter of T, are parallel, as may easily be proved directly. This P-transversal of H is then the directrix of Steiner's parabola.

Again, if Q is the point isogonally conjugate to P with respect to T, since

$$\measuredangle\, A_2A_3A_1 = \measuredangle\, A_2PA_1 = \pi - \measuredangle\, A_1A_2P - \measuredangle\, PA_1A_2 = \pi - \measuredangle\, QA_2A_3 - \measuredangle\, A_3A_1Q,$$

Q is at infinity.[4]

Now $\measuredangle\, A_3A_1Q = \measuredangle\, PA_1A_2 = \measuredangle\, PU_1A_2$. But $\measuredangle\, U_1A_1A_3 = \measuredangle\, U_1A_2A_3$. Moreover, if l is the pedal line, $\measuredangle\, PU_1A_2$ and $\measuredangle\, U_1A_2A_3$ are complementary. Hence A_1Q is perpendicular to A_1U_1. But in this case (in fact for any of Poncelet's lines) A_1U_1 is parallel to l. Hence A_1Q is perpendicular to the pedal line of P.[5] Then PQ is the axis of the Steiner parabola when Q is the point isogonally conjugate to P. It follows that the P-transversal of Q in this case, a tangent to the Steiner parabola at infinity, must be the line at infinity. This can be proved independently by showing that in this case PU_i is parallel to a_i.

4. Let A_{12}, A_{34}; A_{13}, A_{24}; A_{14}, A_{23} (Fig. 2), be opposite vertices of a complete quadrilateral. Let P be its Wallace point,[6] common to the circumcircles of the four triangles T_1, T_2, T_3, T_4, where T_1 is $A_{23}A_{34}A_{24}$. It is evident that the polar reciprocal of the quadrilateral with respect to any circle, center P, is a cyclic quadrangle whose vertices L_1', L_2', L_3', L_4' are the reciprocals of the four sides of the quadrilateral. The circle C', circumscribing the quadrangle, passes through P.

[1] Mackay, *Proc. Edin. Math. Soc.*, 1890–1891.

[2] Boyman (*Archiv der Math. u. Phys.*, 1849, 13, p. 364) gives a theorem equivalent to this corollary.

[3] Steiner, Gergonne's *Annales*, XIX, 1828.

[4] Casey, *Sequel to Euclid*, 1886, Ex. 4, p. 167.

[5] Lachlan, *Modern Pure Geometry*, 1893, Ex. 4, p. 68.

[6] Mackay, *loc. cit.*

Now it is easily proved[1] that the 16 in- and ex-centers of the four triangles formed by taking the vertices of the quadrangle in threes lie four by four on four lines parallel to a bisector of the angle between $L_1'L_3'$ and $L_2'L_4'$ and also four by four on four lines perpendicular to these. In the figure, J_1', J_{21}', J_{34}', J_4' are shown lying on one of these lines.

By reciprocation, remembering the fact asserted in the last paragraph of 2, the 16 P-transversals of the in- and ex-centers of the four triangles T_i ($i = 1, 2, 3, 4$), are concurrent in fours in four points which all lie on a line through P

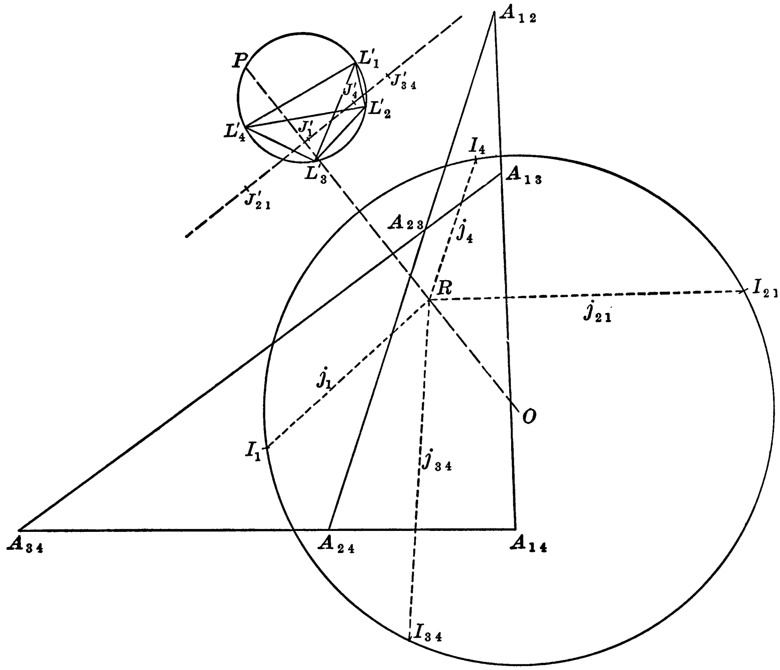

FIG. 2.

bisecting the angle $A_{12}PA_{34}$. One of these points, at which meet the P-transversals of I_1, I_{21}, I_{34}, I_4 is shown in the figure at R.

It may be added that these two lines (PR and the line perpendicular to it at P) are the lines otherwise defined by Steiner,[2] as determined each by the centers of four circles on each of which lie four of the 16 in- and ex-centers of the triangles T. One of these circles, determined by I_1, I_{21}, I_{34}, I_4, Center O, is shown in the figure. The writer has been able to prove this fact only by making use of a long, circuitous proof based on that given by Mention in *Nouv. Ann. de Math.*, 1862, p. 76, proving that the line OP is also a bisector of the angle $A_{12}PA_{34}$. It seems as though a simpler proof might be found.

[1] Neuberg, *Mathesis*, 6, 1906, p. 14. This MONTHLY, 1917, p. 430.

[2] Gergonne's *Annales*, 1828, XVIII, p. 302.